

VIOLATING THE SINGULAR CARDINALS HYPOTHESIS WITHOUT LARGE CARDINALS

BY

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ABSTRACT

We extend a transitive model V of $\text{ZFC} + \text{GCH}$ cardinal preservingly to a model N of $\text{ZF} +$ “GCH holds below \aleph_ω ” + “there is a surjection from the power set of \aleph_ω onto λ ” where λ is an arbitrarily high fixed cardinal in V . The construction can be described as follows: add \aleph_{n+1} many COHEN subsets of \aleph_{n+1} for every $n < \omega$, and adjoin λ many subsets of \aleph_ω which are unions of ω -sequences of those COHEN subsets; then let N be a choiceless submodel generated by equivalence classes of the λ subsets of \aleph_ω modulo an appropriate equivalence relation.

In [1], ARTHUR APTER and the second author constructed a model of $\text{ZF} +$ “GCH holds below \aleph_ω ” + “there is a surjection from $[\aleph_\omega]^\omega$ onto λ ” where λ is an arbitrarily high fixed cardinal in the ground model V . This amounts to a strong *surjective* violation of the *singular cardinals hypothesis* SCH. The construction assumed a measurable cardinal in the ground model. It was also shown in [1] that a measurable cardinal in some inner model is necessary for that combinatorial property, using the DODD-JENSEN covering theorem [2].

In this paper we show that one can work without measurable cardinals if one considers surjections from $\mathcal{P}(\aleph_\omega)$ onto λ . These surjections anyway seem to be more natural than surjections from $[\aleph_\omega]^\omega$ onto λ .

THEOREM 1: *Let V be any ground model of $\text{ZFC} + \text{GCH}$ and let λ be some cardinal in V . Then there is a cardinal preserving model $N \supseteq V$ of the theory $\text{ZF} +$ “GCH holds below \aleph_ω ” + “there is a surjection from $\mathcal{P}(\aleph_\omega)$ onto λ ”.*

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Note that in the presence of the axiom of choice (AC) the latter theory for $\lambda \geq \aleph_{\omega+2}$ has large consistency strength and implies the existence of measurable cardinals of high MITCHELL orders in some inner model (see [3] by the first author). The pcf-theory of SAHARON SHELAH [4] shows that the situation for $\lambda \geq \aleph_{\omega_4}$ is incompatible with AC. Hence Theorem 1 yields a choiceless violation of pcf-theory without the use of large cardinals.

1. The forcing

Fix a ground model V of $\text{ZFC} + \text{GCH}$ and let λ be some regular cardinal in V . We first present two building blocks of our construction. The forcing $P_0 = (P_0, \supseteq, \emptyset)$ adjoins one COHEN subset of \aleph_{n+1} for every $n < \omega$.

$$P_0 = \{p \mid \exists (\delta_n)_{n < \omega} (\forall n < \omega : \delta_n \in [\aleph_n, \aleph_{n+1}) \wedge p : \bigcup_{n < \omega} [\aleph_n, \delta_n) \rightarrow 2)\}.$$

Adjoining one COHEN subset of \aleph_{n+1} for every $n < \omega$ is equivalent to adjoining \aleph_{n+1} -many by the following “two-dimensional” forcing $(P_*, \supseteq, \emptyset)$:

$$P_* = \{p_* \mid \exists (\delta_n)_{n < \omega} (\forall n < \omega : \delta_n \in [\aleph_n, \aleph_{n+1}) \wedge p_* : \bigcup_{n < \omega} [\aleph_n, \delta_n)^2 \rightarrow 2)\}.$$

For $p_* \in P_*$ and $\xi \in [\aleph_0, \aleph_\omega)$ let

$$p_*(\xi) = \{(\zeta, p_*(\xi, \zeta)) \mid (\xi, \zeta) \in \text{dom}(p_*)\}$$

be the ξ -th section of p_* .

LEMMA 1: *Forcing with P_0 (and equivalently P_*) preserves cardinals and the GCH.*

Proof. Consider a V -generic filter G_0 on P_0 . By the GCH in V , $\text{card}^V(P_0) \leq (\text{card}(\mathcal{P}(\aleph_\omega)))^V = \aleph_{\omega+1}^V$. Hence forcing with P_0 preserves all cardinals $> \aleph_{\omega+1}^V$. Every subset $x \subseteq \kappa \in \text{Card}^V$, $x \in V[G_0]$ has a name $\dot{x} \in V$ of the form

$$\dot{x} \subseteq \{\check{\nu} \mid \nu < \kappa\} \times P_0.$$

For $\kappa \geq \aleph_{\omega+1}^V$, the GCH is preserved by

$$(2^\kappa)^{V[G_0]} \leq (\text{card}(\mathcal{P}(\kappa \times P_0)))^V = (\text{card}(\mathcal{P}(\kappa)))^V = (2^\kappa)^V = (\kappa^+)^V.$$

Preservation at cardinals $\aleph_k < \aleph_\omega$ is shown by a product analysis. In V , the forcing P_0 canonically factors into a product

$$P_0 \cong P'_0 \times P''_0$$

with

$$P'_0 = \{p' \mid \exists (\delta_n)_{n < k} (\forall n < k : \delta_n \in [\aleph_n, \aleph_{n+1}) \wedge p' : \bigcup_{n < k} [\aleph_n, \delta_n) \rightarrow 2)\}$$

and

$$P_0'' = \{p'' \mid \exists (\delta_n)_{k \leq n < \omega} (\forall n \in [k, \omega) : \delta_n \in [\aleph_n, \aleph_{n+1}) \wedge p'' : \bigcup_{k \leq n < \omega} [\aleph_n, \delta_n) \rightarrow 2)\}.$$

Let G'_0 and G''_0 the projections of G_0 to P'_0 and P_0'' resp. The forcing P_0'' is $< \aleph_{k+1}$ -closed and hence preserves the power set of \aleph_k . This implies that $\aleph_{i+1}^V = \aleph_{i+1}^{V[G'_0]}$ and $V[G'_0] \models 2^{\aleph_i} = \aleph_{i+1}$ for $i \leq k$. The definition of P'_0 evaluated in the generic extension $V[G'_0]$ yields the original P'_0 . $V[G'_0] \models 2^{\aleph_i} = \aleph_{i+1}$ for $i < k$ implies that $V[G'_0] \models \text{card}(P'_0) \leq \aleph_k$. Thus the extension $V[G''_0][G'_0]$ does not collapse \aleph_{k+1}^V . Every subset $x \subseteq \aleph_k^V$, $x \in V[G_0]$ has a name $\dot{x} \subseteq \{\check{\nu} \mid \nu < \aleph_k^V\} \times P'_0$, $\dot{x} \in V$. Then

$$\begin{aligned} (2^{\aleph_k})^{V[G_0]} &\leq (\text{card}(\mathcal{P}(\{\check{\nu} \mid \nu < \aleph_k^V\} \times P'_0)))^V \leq \\ &\leq (\text{card}(\mathcal{P}(\aleph_k)))^V = (2^{\aleph_k})^V = \aleph_{k+1}^V. \end{aligned}$$

Since $k < \omega$ was arbitrary, $\aleph_{k+1}^V = \aleph_{k+1}^{V[G_0]}$ and $V[G_0] \models 2^{\aleph_k} = \aleph_{k+1}$ for $k < \omega$. Hence $\aleph_\omega^V = \aleph_\omega^{V[G_0]}$.

To bound $(2^{\aleph_\omega})^{V[G_0]}$ observe that in V and in $V[G_0]$

$$\aleph_\omega^{\aleph_0} \leq \aleph_\omega^{\aleph_\omega} = 2^{\aleph_\omega} \leq \prod_{k < \omega} 2^{\aleph_k} \leq \prod_{k < \omega} \aleph_{k+1} \leq \prod_{k < \omega} \aleph_\omega = \aleph_\omega^{\aleph_0}.$$

Since P_0 is $< \aleph_1$ -closed, no new ω -sequences of ordinals are added, and

$$(2^{\aleph_\omega})^{V[G_0]} = (\aleph_\omega^{\aleph_0})^{V[G_0]} \leq (\aleph_\omega^{\aleph_0})^V = (2^{\aleph_\omega})^V = \aleph_{\omega+1}^V.$$

By CANTOR's theorem this also implies that $\aleph_{\omega+1}^V$ is not collapsed, i.e., $\aleph_{\omega+1}^V = \aleph_{\omega+1}^{V[G_0]}$. ■

The forcing employed in the subsequent construction is a kind of finite support product of λ copies of P_0 where the factors are eventually coupled via P_* .

DEFINITION 1: Define the forcing (P, \leq_P, \emptyset) by:

$$\begin{aligned} P &= \{(p_*, (a_i, p_i)_{i < \lambda}) \mid \\ &\quad \exists (\delta_n)_{n < \omega} \exists D \in [\lambda]^{< \omega} (\forall n < \omega : \delta_n \in [\aleph_n, \aleph_{n+1}), \\ &\quad p_* : \bigcup_{n < \omega} [\aleph_n, \delta_n)^2 \rightarrow 2, \\ &\quad \forall i \in D : p_i : \bigcup_{n < \omega} [\aleph_n, \delta_n) \rightarrow 2 \wedge p_i \neq \emptyset, \\ &\quad \forall i \in D : a_i \in [\aleph_\omega \setminus \aleph_0]^{< \omega} \wedge \forall n < \omega : \text{card}(a_i \cap [\aleph_n, \aleph_{n+1})) \leq 1, \\ &\quad \forall i \notin D (a_i = p_i = \emptyset)\}. \end{aligned}$$

If $p = (p_*, (a_i, p_i)_{i < \lambda}) \in P$ then $p \in P_*$ and $p_i \in P_0$, with all but finitely many p_i being \emptyset . Extending p_i is controlled by linking ordinals $\xi \in a_i$. More specifically extending p_i in the interval $[\aleph_n, \aleph_{n+1})$ is controlled by $\xi \in a_i \cap [\aleph_n, \aleph_{n+1})$ if that intersection is nonempty. Let $\text{supp}(p) = \{i < \lambda \mid p_i \neq \emptyset\}$ be the support of $p = (p_*, (a_i, p_i)_{i < \lambda})$, i.e., the set D in the definition of P . P is partially ordered by

$$p' = (p'_*, (a'_i, p'_i)_{i < \lambda}) \leq_P (p_*, (a_i, p_i)_{i < \lambda}) = p$$

iff

$$a) \ p'_* \supseteq p_*, \forall i < \lambda (a'_i \supseteq a_i \wedge p'_i \supseteq p_i),$$

b) (Linking property)

$$\forall i < \lambda \forall n < \omega \forall \xi \in a_i \cap [\aleph_n, \aleph_{n+1}) \forall \zeta \in \text{dom}(p'_i \setminus p_i) \cap [\aleph_n, \aleph_{n+1}) :$$

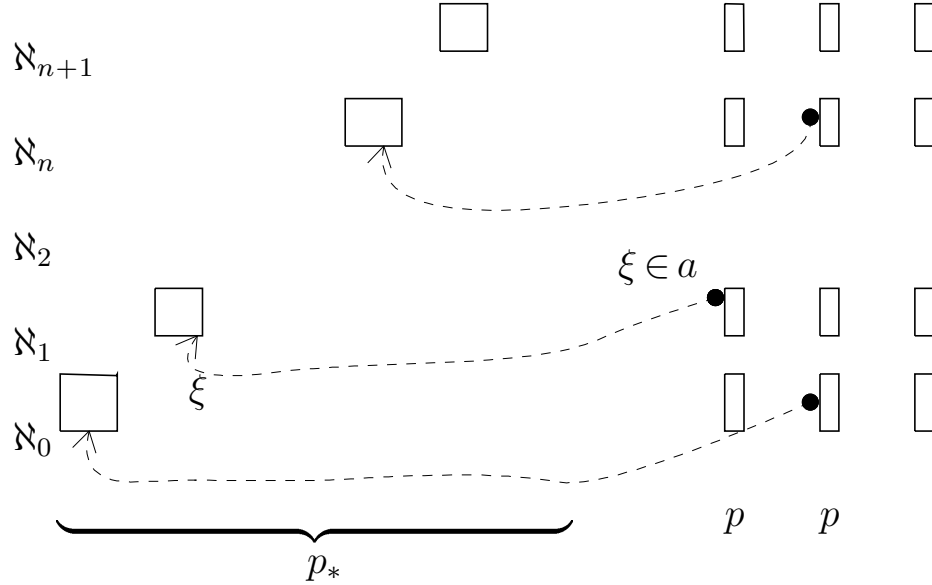
$$p'_i(\zeta) = p'_*(\xi)(\zeta),$$

c) (Independence property)

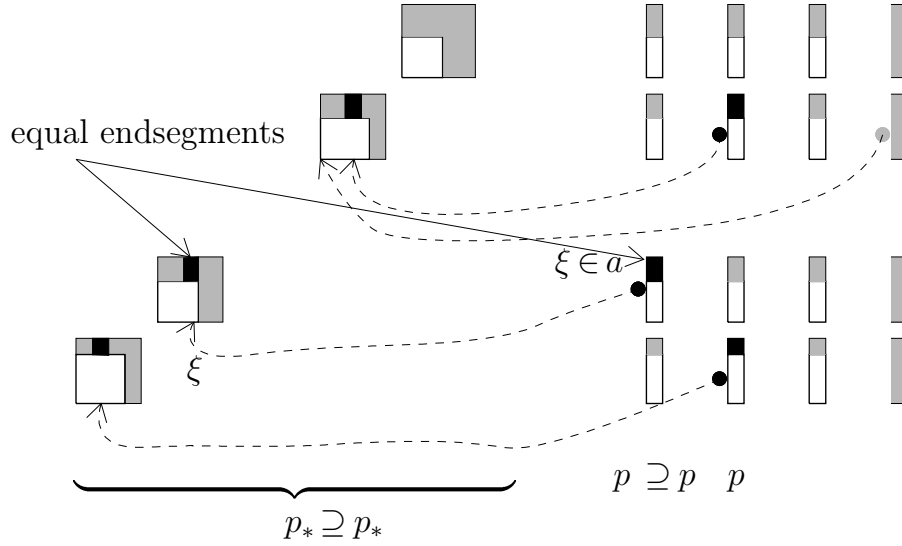
$$\forall j \in \text{supp}(p) : (a'_j \setminus a_j) \cap \bigcup_{i \in \text{supp}(p), i \neq j} a'_i = \emptyset.$$

$1 = (\emptyset, (\emptyset, \emptyset)_{i < \lambda})$ is the maximal element of P .

One may picture a condition $p \in P$ as



and an extension $(p'_*, (a'_i, p'_i)_{i < \lambda}) \leq_P (p_*, (a_i, p_i)_{i < \lambda})$ as



The gray areas indicate new 0-1-values in the extension $p' \leq_P p$, and the black areas indicate equality of new values forced by linking ordinals ξ .

Let G be a V -generic filter for P . Several generic objects can be extracted from G . It is easy to see that the set

$$G_* = \{p_* \in P_* \mid (p_*, (a_i, p_i)_{i < \lambda}) \in G\}$$

is V -generic for the partial order P_* . Set

$$A_* = \bigcup_{n < \omega} G_* : \bigcup_{n < \omega} [\aleph_n, \aleph_{n+1})^2 \rightarrow 2.$$

For $\xi \in [\aleph_n, \aleph_{n+1})$ let

$$A_*(\xi) = \{(\zeta, A_*(\xi, \zeta)) \mid \zeta \in [\aleph_n, \aleph_{n+1})\} : [\aleph_n, \aleph_{n+1}) \rightarrow 2$$

be the (characteristic function of the) ξ -th new COHEN subset of \aleph_{n+1} in the generic extension.

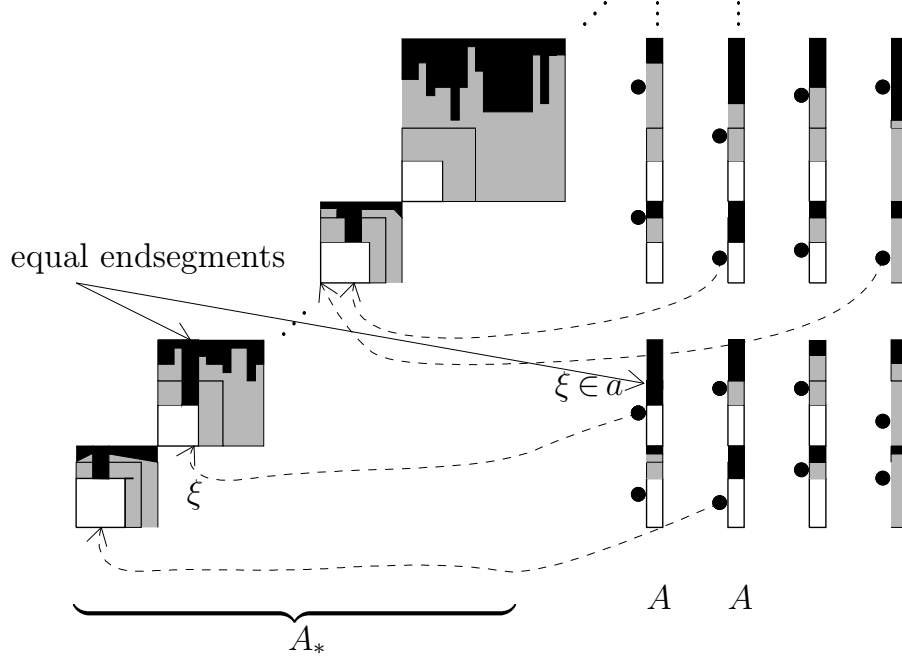
For $i < \lambda$ let

$$A_i = \bigcup \{p_i \mid (p_*, (a_j, p_j)_{j < \lambda}) \in G\} : [\aleph_0, \aleph_\omega) \rightarrow 2$$

be the (characteristic function of the) i -th subset of \aleph_ω adjoined by the forcing P . A_i is V -generic for the forcing P_0 .

By the linking property *b*) of Definition 1, on a final segment, the characteristic functions $A_i \upharpoonright [\aleph_n, \aleph_{n+1})$ will be equal to some $A_*(\xi)$. The independence property *c*) ensures that sets $A_i, A_j \subseteq \aleph_\omega$ with $i \neq j$ correspond to eventually disjoint, “parallel” paths through the forcing P_* .

The generic filter and the extracted generic objects may be pictured as follows. Black colour indicates agreement between parts of the A_i and of A_* ; for each $i < \lambda$, some endsegment of $A_i \cap \aleph_{n+1}$ occurs as an endsegment of some vertical cut in $A_* \cap \aleph_{n+1}^2$.



LEMMA 2: P satisfies the $\aleph_{\omega+2}$ -chain condition.

Proof. Let $\{(p_*^j, (a_i^j, p_i^j)_{i < \lambda}) \mid j < \aleph_{\omega+2}\} \subseteq P$. We shall show that two elements of the sequence are compatible. Since

$$\text{card}(P_*) = \text{card}(P_0) \leq 2^{\aleph_\omega} = \aleph_{\omega+1}$$

we may assume that there is $p_* \in P_*$ such that $\forall j < \aleph_{\omega+2} : p_*^j = p_*$. We may assume that the supports $\text{supp}((p_*, (a_i^j, p_i^j)_{i < \lambda})) \subseteq \lambda$ form a Δ -system with a finite kernel $I \subseteq \lambda$. Finally we may assume that there are $(a_i, p_i)_{i \in I}$ such that $\forall j < \aleph_{\omega+2} \forall i \in I : (a_i^j, p_i^j) = (a_i, p_i)$. Then $(p_*^0, (a_i^0, p_i^0)_{i < \lambda})$ and $(p_*^1, (a_i^1, p_i^1)_{i < \lambda})$ are compatible since

$$(p_*, (a_i^0 \cup a_i^1, p_i^0 \cup p_i^1)_{i < \lambda}) \leq_P (p_*^0, (a_i^0, p_i^0)_{i < \lambda})$$

and

$$(p_*, (a_i^0 \cup a_i^1, p_i^0 \cup p_i^1)_{i < \lambda}) \leq_P (p_*^1, (a_i^1, p_i^1)_{i < \lambda}).$$

■

By Lemma 2, cardinals $\geq \aleph_{\omega+2}^V$ are absolute between V and $V[G]$.

2. Fuzzifying the A_i

We want to construct a model which contains all the A_i and a map which maps every A_i to its index i . An *injective* map $\lambda \mapsto \mathcal{P}(\aleph_\omega)$ for some high λ would imply large consistency strength (see [1]). To disallow such maps, the A_i are replaced by their equivalence classes modulo an appropriate equivalence relation.

The *exclusive or* function $\oplus : 2 \times 2 \rightarrow 2$ is defined by

$$a \oplus b = 0 \text{ iff } a = b.$$

Obviously, $(a \oplus b) \oplus (b \oplus c) = a \oplus c$. For functions

$$A, A' : \text{dom}(A) = \text{dom}(A') \rightarrow 2$$

define the *pointwise exclusive or* $A \oplus A' : \text{dom}(A) \rightarrow 2$ by

$$(A \oplus A')(\xi) = A(\xi) \oplus A'(\xi).$$

For functions $A, A' : (\aleph_\omega \setminus \aleph_0) \rightarrow 2$ define an equivalence relation \sim by $A \sim A'$ iff

$$\exists n < \omega ((A \oplus A') \upharpoonright \aleph_{n+1} \in V[G_*] \wedge (A \oplus A') \upharpoonright [\aleph_{n+1}, \aleph_\omega) \in V).$$

This relation is clearly reflexive and symmetric. We show transitivity. Consider $A \sim A' \sim A''$. Choose $n < \omega$ such that

$$(A \oplus A') \upharpoonright \aleph_{n+1} \in V[G_*] \wedge (A \oplus A') \upharpoonright [\aleph_{n+1}, \aleph_\omega) \in V$$

and

$$(A' \oplus A'') \upharpoonright \aleph_{n+1} \in V[G_*] \wedge (A' \oplus A'') \upharpoonright [\aleph_{n+1}, \aleph_\omega) \in V.$$

Then

$$(A \oplus A'') \upharpoonright \aleph_{n+1} = ((A \oplus A') \upharpoonright \aleph_{n+1} \oplus (A' \oplus A'') \upharpoonright \aleph_{n+1}) \in V[G_*]$$

and

$$\begin{aligned} (A \oplus A'') \upharpoonright [\aleph_{n+1}, \aleph_\omega) &= \\ &= ((A \oplus A') \upharpoonright [\aleph_{n+1}, \aleph_\omega) \oplus (A' \oplus A'') \upharpoonright [\aleph_{n+1}, \aleph_\omega)) \in V. \end{aligned}$$

Hence $A \sim A''$.

For $A : (\aleph_\omega \setminus \aleph_0) \rightarrow 2$ define the \sim -equivalence class

$$\tilde{A} = \{A' \mid A' \sim A\}.$$

3. The symmetric submodel

Our final model will be a model generated by the following parameters and their constituents

- $T_* = \mathcal{P}(< \kappa)^{V[A_*]}$, setting $\kappa = \aleph_\omega^V$;
- $\vec{A} = (\tilde{A}_i \mid i < \lambda)$.

The model

$$N = \text{HOD}^{V[G]}(V \cup \{T_*, \vec{A}\} \cup T_* \cup \bigcup_{i < \lambda} \tilde{A}_i)$$

consists of all sets which, in $V[G]$ are hereditarily definable from parameters in the transitive closure of $V \cup \{T_*, \vec{A}\}$. This model is *symmetric* in the sense that it is generated from parameters which are invariant under certain (partial) isomorphisms of the forcing P .

LEMMA 3: *N is a model of ZF, and there is a surjection $f : \mathcal{P}(\kappa) \rightarrow \lambda$ in N .*

Proof. Note that for every $i < \lambda$: $A_i \in N$.

(1) Let $i < j < \lambda$. Then $A_i \approx A_j$.

Proof. Assume instead that $A_i \sim A_j$. Then take $n < \omega$ such that $v = (A_i \oplus A_j) \restriction [\aleph_{n+1}, \aleph_\omega) \in V$.

The set

$$D = \{ (p_*, (a_k, p_k)_{k < \lambda}) \mid \exists \xi \in [\aleph_{n+1}, \aleph_\omega) \\ (\xi \in \text{dom}(p_i) \cap \text{dom}(p_j) \wedge v(\xi) \neq p_i(\xi) \oplus p_j(\xi)) \} \in V$$

is readily seen to be dense in P . Take $(p_*, (a_k, p_k)_{k < \lambda}) \in D \cap G$. Take $\xi \in [\aleph_{n+1}, \aleph_\omega)$ such that

$$\xi \in \text{dom}(p_i) \cap \text{dom}(p_j) \wedge v(\xi) \neq p_i(\xi) \oplus p_j(\xi).$$

Since $p_i \subseteq A_i$ and $p_j \subseteq A_j$ we have $v(\xi) \neq A_i(\xi) \oplus A_j(\xi)$ and $v \neq (A_i \oplus A_j) \restriction [\aleph_{n+1}, \aleph_\omega)$. Contradiction. *qed*(1)

Thus

$$f(z) = \begin{cases} i, & \text{if } z \in \tilde{A}_i; \\ 0, & \text{else;} \end{cases}$$

is a well-defined surjection $f : \mathcal{P}(\kappa) \rightarrow \lambda$, and f is definable in N from the parameters κ and \vec{A} . ■

The main theorem will be established by showing that, in N , the situation below κ is largely as in V , in particular $\kappa = \aleph_\omega^N$. This requires an analysis of sets of ordinals in N .

LEMMA 4: *Every set $X \in N$ is definable in $V[G]$ in the following form: there are an \in -formula φ , $x \in V$, $n < \omega$, and $i_0, \dots, i_{l-1} < \lambda$ such that*

$$X = \{u \in V[G] \mid V[G] \models \varphi(u, x, T_*, \vec{A}, A_* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}})\}.$$

Proof. By the original definition, every set in N is definable in $V[G]$ from finitely many parameters in

$$V \cup \{T_*, \vec{A}\} \cup T_* \cup \bigcup_{i < \lambda} \tilde{A}_i.$$

To reduce the class of defining parameters to

$$V \cup \{T_*, \vec{A}\} \cup \{A_* \restriction (\aleph_{n+1}^V)^2 \mid n < \omega\} \cup \{A_i \mid i < \lambda\}$$

observe:

- Let $x \in T_*$ be a bounded subset of \aleph_ω^V with $x \in V[A_*]$. A standard product analysis of the generic extension $V[G_*] = V[A_*]$ of V yields that $x \in V[A_* \restriction (\aleph_{n+1}^V)^2]$ for some $n < \omega$.

– Let $y \in \tilde{A}_i$. Then $y \sim A_i$, i.e.,

$$(y \oplus A_i) \upharpoonright \aleph_{m+1} \in V[G_*] \wedge (y \oplus A_i) \upharpoonright [\aleph_{m+1}, \aleph_\omega) \in V$$

for some $m < \omega$. Let $z = (y \oplus A_i) \upharpoonright \aleph_{m+1} \in V[A_*]$. By the previous argument $z \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2]$ for some $n < \omega$. Let $z' = (y \oplus A_i) \upharpoonright [\aleph_{m+1}, \aleph_\omega) \in V$. Then

$$\begin{aligned} y &= (y \upharpoonright \aleph_{m+1}) \cup (y \upharpoonright [\aleph_{m+1}, \aleph_\omega)) \\ &= ((z \oplus A_i) \upharpoonright \aleph_{m+1}) \cup ((z' \oplus A_i) \upharpoonright [\aleph_{m+1}, \aleph_\omega)) \\ &\in V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_i]. \end{aligned}$$

Finitely many parameters of the form $A_* \upharpoonright (\aleph_{n+1}^V)^2$ can then be incorporated into a single such parameter taking a sufficiently high $n < \omega$. ■

4. Approximating N

Concerning sets of ordinals, the model N can be approximated by “mild” generic extensions of the ground model. Note that many set theoretic notions only refer to ordinals and sets of ordinals.

LEMMA 5: *Let $X \in N$ and $X \subseteq \text{Ord}$. Then there are $n < \omega$ and $i_0, \dots, i_{l-1} < \lambda$ such that*

$$X \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}].$$

Proof. By Lemma 4 take an \in -formula φ , $x \in V$, $n < \omega$, and $i_0, \dots, i_{l-1} < \lambda$ such that

$$X = \{u \in \text{Ord} \mid V[G] \models \varphi(u, x, T_*, \vec{A}, A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}})\}.$$

By taking n sufficiently large, we may assume that

$$\forall j < k < l \forall m \in [n, \omega) \forall \delta \in [\aleph_m, \aleph_{m+1}) :$$

$$A_{i_j} \upharpoonright [\delta, \aleph_{m+1}) \neq A_{i_k} \upharpoonright [\delta, \aleph_{m+1}).$$

For $j < l$ set

$$a_{i_j}^* = \{\xi \mid \exists m \leq n \quad \exists \delta \in [\aleph_m, \aleph_{m+1}) :$$

$$A_{i_j} \upharpoonright [\delta, \aleph_{m+1}) = A_*(\xi) \upharpoonright [\delta, \aleph_{m+1})\}$$

where $A_*(\xi) = \{(\zeta, A_*(\xi, \zeta)) \mid (\xi, \zeta) \in \text{dom}(A_*)\}$. By the properties of Q , $a_{i_j}^* \subseteq \aleph_{n+1}$ is finite and $\forall m \leq n : \text{card}(a_{i_j}^* \cap [\aleph_m, \aleph_{m+1})) = 1$.

Now define

$$X' = \{u \in \text{Ord} \mid \text{there is } p = (p_*, (a_i, p_i)_{i < \lambda}) \in P \text{ such that}$$

$$p_* \upharpoonright (\aleph_{n+1}^V)^2 \subseteq A_* \upharpoonright (\aleph_{n+1}^V)^2,$$

$$a_{i_0} \supseteq a_{i_0}^*, \dots, a_{i_{l-1}} \supseteq a_{i_{l-1}}^*,$$

$$p_{i_0} \subseteq A_{i_0}, \dots, p_{i_{l-1}} \subseteq A_{i_{l-1}}, \text{ and}$$

$$p \Vdash \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \upharpoonright (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \dots, \dot{A}_{i_{l-1}})\},$$

where $\sigma, \tau, \dot{A}, \dot{A}_{i_0}, \dots, \dot{A}_{i_{l-1}}$ are canonical names for

$$T_*, \vec{A}, A_*, A_{i_0}, \dots, A_{i_{l-1}}$$

resp.

Then $X' \in V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}]$.

(1) $X \subseteq X'$.

Proof. Consider $u \in X$. Take $p = (p_*, (a_i, p_i)_{i < \lambda}) \in G$ such that

$$p \Vdash \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \upharpoonright (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \dots, \dot{A}_{i_{l-1}}).$$

Then $p_* \upharpoonright (\aleph_{n+1}^V)^2 \subseteq A_* \upharpoonright (\aleph_{n+1}^V)^2$ and $p_{i_0} \subseteq A_{i_0}, \dots, p_{i_{l-1}} \subseteq A_{i_{l-1}}$. Using a density argument we may also assume that $\text{card}(a_{i_0} \cap \aleph_{n+1}) = \dots = \text{card}(a_{i_{l-1}} \cap \aleph_{n+1}) = n$. Then $a_{i_0} \supseteq a_{i_0}^*, \dots, a_{i_{l-1}} \supseteq a_{i_{l-1}}^*$. Thus $u \in X'$. *qed*(1)

The converse direction, $X' \subseteq X$, is more involved and uses an isomorphism argument. Suppose for a contradiction that there were $u \in X' \setminus X$. Then take a condition $p = (p_*, (a_i, p_i)_{i < \lambda}) \in P$ as in the definition of X' , i.e.,

- (2) $p_* \upharpoonright (\aleph_{n+1}^V)^2 \subseteq A_* \upharpoonright (\aleph_{n+1}^V)^2$,
- (3) $a_{i_0} \supseteq a_{i_0}^*, \dots, a_{i_{l-1}} \supseteq a_{i_{l-1}}^*$,
- (4) $p_{i_0} \subseteq A_{i_0}, \dots, p_{i_{l-1}} \subseteq A_{i_{l-1}}$, and
- (5) $p \Vdash \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \upharpoonright (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \dots, \dot{A}_{i_{l-1}})$.

By $u \notin X$ take $p' = (p'_*, (a'_i, p'_i)_{i < \lambda}) \in G$ such that

- (6) $p' \Vdash \neg \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \upharpoonright (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \dots, \dot{A}_{i_{l-1}})$.

By genericity we may assume that

- (7) $p'_* \upharpoonright (\aleph_{n+1}^V)^2 \subseteq A_* \upharpoonright (\aleph_{n+1}^V)^2$
- (8) $a'_{i_0} \supseteq a_{i_0}^*, \dots, a'_{i_{l-1}} \supseteq a_{i_{l-1}}^*$, and
- (9) $p'_{i_0} \subseteq A_{i_0}, \dots, p'_{i_{l-1}} \subseteq A_{i_{l-1}}$.

By strengthening the conditions we can arrange that p and p' have similar “shapes” whilst preserving the above conditions (2) to (9):

- (10) ensure that $\text{supp}(p) = \text{supp}(p')$; choose some \aleph_{m+1} such that $\forall i \in \text{supp}(p)(a_i \subseteq \aleph_{m+1} \wedge a'_i \subseteq \aleph_{m+1})$;

- (11) extend the a_i and a'_i such that

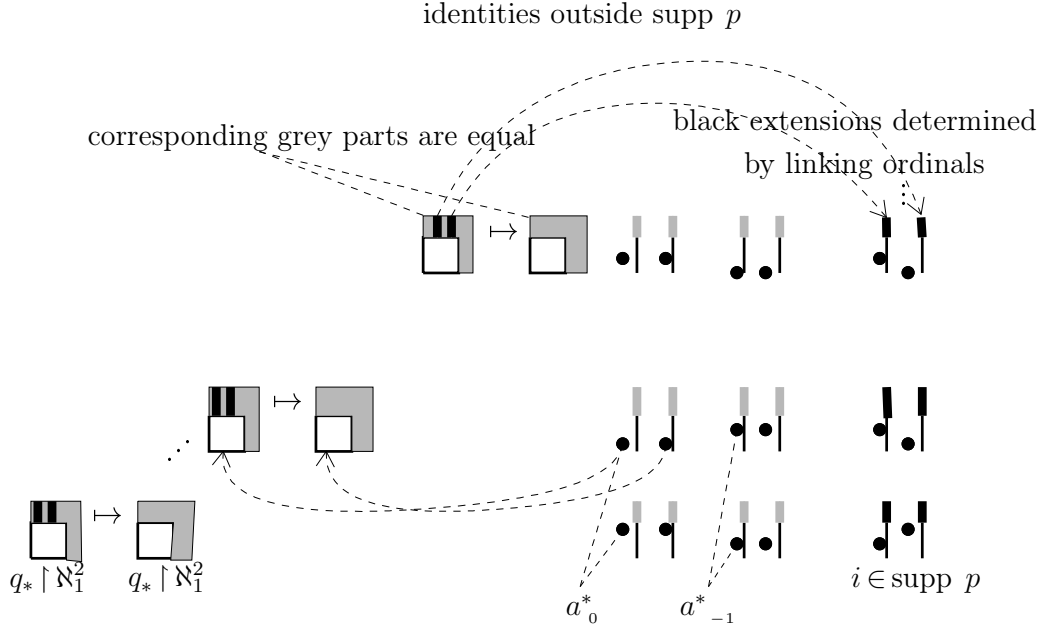
$$\forall i \in \text{supp}(p) \forall k \leq m : \text{card}(a_i \cap [\aleph_k, \aleph_{k+1})) = \text{card}(a'_i \cap [\aleph_k, \aleph_{k+1})) = 1;$$

- (12) also extend the conditions so that they involve the same “linking” ordinals, possibly at different positions within the conditions:

$$\bigcup_{i < \lambda} a_i = \bigcup_{i < \lambda} a'_i$$

- (13) extend the p_* and p_i 's in p and p' resp. so that for some sequence $(\delta_k | k < \omega)$:

$$\text{dom}(p_*) = \text{dom}(p'_*) = \bigcup_{k < \omega} [\aleph_k, \delta_k)^2$$



We verify that $\pi : (P \restriction p, \leq_P) \rightarrow (P \restriction p', \leq_P)$ is an *isomorphism*.

$$(17) (q'_*, (b'_i, q'_i)_{i < \lambda}) \in P,$$

since it has the same structure as $(q_*, (b_i, q_i)_{i < \lambda})$, with some function values altered.

$$(18) (q'_*, (b'_i, q'_i)_{i < \lambda}) \leq_P (p'_*, (a'_i, p'_i)_{i < \lambda}).$$

Proof. $q'_* \supseteq p'_*$ since $q'_* = (q_* \setminus p_*) \cup p'_*$, see (14). Similarly we get $b'_i \supseteq a'_i$ and $q'_i \supseteq p'_i$. To check the linking property (Definition 1, b)), consider $i < \lambda$, $n < \omega$, and $\xi' \in a'_i \cap [\aleph_n, \aleph_{n+1})$. For $\zeta \in \text{dom}(q'_i \setminus p'_i)$ we have

$$q'_i(\zeta) = q_*(\xi', \zeta) = q'_*(\xi', \zeta).$$

Finally we have to show the independence property (Definition 1, c)) within the linking ordinals. Consider $j \in \text{supp}(p') = \text{supp}(p)$. We claim that $(b'_j \setminus a'_j) \cap \bigcup_{i \in \text{supp}(p'), i \neq j} b'_i = \emptyset$. Assume for a contradiction that $\xi' \in (b'_j \setminus a'_j) \cap b'_i$ for some $i \in \text{supp}(p')$, $i \neq j$. Then $\xi' \in (b_j \setminus a_j) \cap ((b_i \setminus a_i) \cup a'_i)$. The case $\xi' \in (b_j \setminus a_j) \cap (b_i \setminus a_i)$ is impossible by the independence property in $q \leq_P p$. And

$$(b_j \setminus a_j) \cap \bigcup_{i \in \text{supp}(p'), i \neq j} a'_i = (b_j \setminus a_j) \cap \bigcup_{i \in \text{supp}(p), i \neq j} a_i = \emptyset$$

by the independence property in $q \leq_P p$ and by (12). qed(18)

(19) π is order-preserving.

Proof. Consider

$$r = (r_*, (c_i, r_i)_{i < \lambda}) \leq_P q = (q_*, (b_i, q_i)_{i < \lambda}) \leq_P p = (p_*, (a_i, p_i)_{i < \lambda})$$

and $\pi(r) = r' = (r'_*, (c'_i, r'_i)_{i < \lambda})$, $\pi(q) = q' = (q'_*, (b'_i, q'_i)_{i < \lambda})$. We show that $r' \leq_P q'$. Concerning the inclusions:

- $r'_* = (r_* \setminus p_*) \cup p'_* \supseteq (q_* \setminus p_*) \cup p'_* = q'_*$;
- $c'_i = (c_i \setminus a_i) \cup a'_i \supseteq (b_i \setminus a_i) \cup a'_i = b'_i$;
- if $i \in \lambda \setminus \text{supp}(q)$, then $q'_i = \emptyset$ and hence $q'_i \subseteq r'_i$. If $i \in \text{supp}(q)$ then $i \in \text{supp}(r)$, and $\text{dom}(r'_i) = \text{dom}(r_i)$ and $\text{dom}(q'_i) = \text{dom}(q_i)$. So we have

$$\text{dom}(p_i) = \text{dom}(p'_i) \subseteq \text{dom}(q'_i) \subseteq \text{dom}(r'_i).$$

For $\zeta \in \text{dom}(q'_i)$ we have to show that $q'_i(\zeta) = r'_i(\zeta)$. In case $\zeta \in \text{dom}(p_i)$ we have

$$q'_i(\zeta) = p'_i(\zeta) = r'_i(\zeta).$$

In case $\zeta \notin \text{dom}(p_i) \wedge \exists k < \omega : \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a'_i \cap [\aleph_k, \aleph_{k+1}) = \{\xi'\}$ we have

$$q'_i(\zeta) = q_*(\xi', \zeta) = r_*(\xi', \zeta) = r'_i(\zeta).$$

In case $\zeta \in \text{dom}(p_i) \wedge \exists k < \omega : \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a'_i \cap [\aleph_k, \aleph_{k+1}) = \emptyset$ we have

$$q'_i(\zeta) = q_i(\zeta) = r_i(\zeta) = r'_i(\zeta).$$

For the linking property consider $i < \lambda$, $n < \omega$, and $\xi' \in b'_i \cap [\aleph_n, \aleph_{n+1})$. We have to check that $\forall \zeta \in \text{dom}(r'_i \setminus q'_i) \cap [\aleph_n, \aleph_{n+1}) : r'_i(\zeta) = r'_*(\xi', \zeta)$. So consider $\zeta \in \text{dom}(r'_i \setminus q'_i) \cap [\aleph_n, \aleph_{n+1})$. Note that $b'_i = (b_i \setminus a_i) \cup a'_i$. In case $\xi' \in a'_i$ we get:

$$r'_i(\zeta) = r_*(\xi', \zeta) = r'_*(\xi', \zeta).$$

If $\xi' \in b_i \setminus a_i$, $\xi' \notin a'_i$ and so $a'_i \cap [\aleph_n, \aleph_{n+1}) = \emptyset$. Hence

$$r'_i(\zeta) = r_i(\zeta) = r_*(\xi', \zeta) = r'_*(\xi', \zeta).$$

For the independence property consider $j \in \text{supp}(q')$. We have to show that

$$(c'_j \setminus b'_j) \cap \bigcup_{i \in \text{supp}(q'), i \neq j} c'_i = \emptyset.$$

Suppose for a contradiction that $\xi' \in (c'_j \setminus b'_j) \cap \bigcup_{i \in \text{supp}(q'), i \neq j} c'_i$. Then $\xi' \in c'_j \setminus b'_j = c_j \setminus b_j$. Take $i \in \text{supp}(q'), i \neq j$ such that $\xi' \in c'_i$. If $i \in \text{supp}(p')$ this contradicts the property $r' \leq_P p'$. So $i \in \text{supp}(q') \setminus \text{supp}(p')$. Then $c'_i = c_i$ and $\xi' \in (c_j \setminus b_j) \cap c_i$. But this contradicts the independence property for $r \leq_P q$. *qed*(19)

The definition of the map π only uses properties of p and p' which are the same for both of p and p' . So we can similarly define a map

$$\pi' : (P \upharpoonright p', \leq_P) \rightarrow (P \upharpoonright p, \leq_P),$$

where for $q' = (q'_*, (b'_i, q'_i)_{i < \lambda}) \leq_P (p'_*, (a'_i, p'_i)_{i < \lambda}) = p'$ the image $\pi'(q') = (q_*, (b_i, q_i)_{i < \lambda})$ is defined by

$$(20) \quad q_* = (q'_* \setminus p'_*) \cup p_*;$$

$$(21) \quad \text{for } i < \lambda \text{ let } b_i = (b'_i \setminus a'_i) \cup a_i;$$

(22) for $i \in \lambda \setminus \text{supp}(q')$ let $q_i = \emptyset$, and for $i \in \text{supp}(q')$ define $q_i : \text{dom}(q'_i) \rightarrow 2$ by setting $q_i(\zeta)$ equal to

$$\begin{cases} p_i(\zeta), & \text{if } \zeta \in \text{dom}(p'_i), \\ q'_*(\xi, \zeta), & \text{if } \zeta \notin \text{dom}(p'_i) \wedge \\ & \wedge \exists k < \omega : \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a_i \cap [\aleph_k, \aleph_{k+1}) = \{\xi\}, \\ q'_i(\zeta), & \text{if } \zeta \notin \text{dom}(p'_i) \wedge \\ & \wedge \exists k < \omega : \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a_i \cap [\aleph_k, \aleph_{k+1}) = \emptyset. \end{cases}$$

The maps π and π' are inverses:

(23) $\pi' \circ \pi : (P \restriction p, \leq_P) \rightarrow (P \restriction p, \leq_P)$ is the identity on $(P \restriction p, \leq_P)$.

Proof. Let $(q_*, (b_i, q_i)_{i < \lambda}) \leq_P p = (p_*, (a_i, p_i)_{i < \lambda})$ and let

$$\pi(q_*, (b_i, q_i)_{i < \lambda}) = (q'_*, (b'_i, q'_i)_{i < \lambda}).$$

Concerning the first component,

$$q_* \xrightarrow{\pi} (q_* \setminus p_*) \cup p'_* \xrightarrow{\pi'} (((q_* \setminus p_*) \cup p'_*) \setminus p'_*) \cup p_* = q_*.$$

For $i < \lambda$,

$$b_i \xrightarrow{\pi} (b_i \setminus a_i) \cup a'_i \xrightarrow{\pi'} (((b_i \setminus a_i) \cup a'_i) \setminus a'_i) \cup a_i = b_i.$$

For $i \in \lambda \setminus \text{supp}(q)$, $q_i = q'_i = \emptyset$ and so

$$q_i \xrightarrow{\pi} q'_i \xrightarrow{\pi'} q_i.$$

Now consider $i \in \text{supp}(q) = \text{supp}(q')$. Then $\text{dom}(q_i) = \text{dom}(q'_i)$. Let $\zeta \in \text{dom}(q_i)$. In case $\zeta \in \text{dom}(p_i) = \text{dom}(p'_i)$ we have

$$q_i(\zeta) = p_i(\zeta) \xrightarrow{\pi} p'_i(\zeta) = q'_i(\zeta) \xrightarrow{\pi'} p_i(\zeta) = q_i(\zeta).$$

In case $\zeta \notin \text{dom}(p_i) \wedge \exists k < \omega : \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a_i \cap [\aleph_k, \aleph_{k+1}) = \{\xi\}$ let $a'_i \cap [\aleph_k, \aleph_{k+1}) = \{\xi'\}$. Then $q_i(\zeta) = q_*(\xi, \zeta)$ and $q'_i(\zeta) = q'_*(\xi', \zeta)$. Then

$$q_i(\zeta) \xrightarrow{\pi} q'_*(\xi', \zeta) \xrightarrow{\pi'} q_*(\xi, \zeta) = q_i(\zeta).$$

Finally, if $\zeta \notin \text{dom}(p_i) \wedge \exists k < \omega : \zeta \in [\aleph_k, \aleph_{k+1}) \wedge a_i \cap [\aleph_k, \aleph_{k+1}) = \emptyset$ then $q_i(\zeta) = q_i(\zeta)$ and so

$$q_i(\zeta) \xrightarrow{\pi} q'_i(\zeta) \xrightarrow{\pi'} q_i(\zeta).$$

Thus

$$p \xrightarrow{\pi} p' \xrightarrow{\pi'} p.$$

qed(23)

Similarly,

(24) $\pi \circ \pi' : (P' \restriction p', \leq_P) \rightarrow (P' \restriction p', \leq_P)$ is the identity on $(P' \restriction p', \leq_P)$.

Hence $\pi : (P \restriction p, \leq_P) \rightarrow (P \restriction p', \leq_P)$ is an isomorphism. Before we apply π to generic filters and objects defined from them, we note some properties of π .

(25) Let $q = (q_*, (b_i, q_i)_{i < \lambda}) \leq_P p$ and $\pi(q) = (q'_*, (b'_i, q'_i)_{i < \lambda})$. Then $q'_* \restriction (\aleph_{n+1}^V)^2 = q_* \restriction (\aleph_{n+1}^V)^2$, and $q'_{i_0} = q_{i_0}, \dots, q'_{i_{l-1}} = q_{i_{l-1}}$.

Now let H_0 be a V -generic filter for $(P \restriction p, \leq_P)$ with $p \in H_0$. Then

$$H = \{r \in P \mid \exists q \in H_0 : q \leq_P r\}$$

is a V -generic filter for P with $p \in H$.

Moreover, $H'_0 = \pi[H_0]$ is a V -generic filter for $(P \restriction p', \leq_P)$ with $p' \in H'_0$ and

$$H' = \{r \in P \mid \exists q \in H'_0 : q \leq_P r\}$$

is a V -generic filter for P with $p' \in H'$.

(26) $V[H] = V[H']$ since the generic filters can be defined from each other using the isomorphism $\pi \in V$.

Now define the parameters used in the definition of the model N from the generic filters H and H' :

$$H_* = \{q_* \in P_* \mid (q_*, (b_i, q_i)_{i < \lambda}) \in H\}, T_* = \sigma^H, \vec{A} = \tau^H, A_* = \dot{A}^H, \text{ and } A_i = \dot{A}_i^H \text{ for } i < \lambda$$

and

$$H'_* = \{q_* \in P_* \mid (q_*, (b_i, q_i)_{i < \lambda}) \in H'\}, T'_* = \sigma^{H'}, \vec{A}' = \tau^{H'}, A'_* = \dot{A}^{H'}, \text{ and } A'_i = \dot{A}_i^{H'} \text{ for } i < \lambda.$$

where $\sigma, \tau, \dot{A}, \dot{A}_i$ are the canonical names for T_*, \vec{A}, A_*, A_i resp. used in the definition of X' above. Note that to simplify notation we are redefining the previously used constants T_*, \vec{A}, A_*, A_i for the remainder of the current proof. This does not conflict with the use of these constants before and after this proof.

$$(27) \ V[H_*] = V[H'_*].$$

Proof. Since H_* is V -generic for P_* and $p_* \in P_*$, $H_* \cap \{q_* \in P_* \mid q_* \supseteq p_*\}$ is, over the ground model V , equidefinable with H_* . Hence

$$\begin{aligned} V[H_*] &= V[H_* \cap \{q_* \in P_* \mid q_* \supseteq p_*\}] \\ &\supseteq V[\{(q_* \setminus p_*) \cup p'_* \mid q_* \in H_* \cap \{q_* \in P_* \mid q_* \supseteq p_*\}\}] \\ &= V[H'_* \cap \{q_* \in P_* \mid q_* \supseteq p'_*\}] \\ &= V[H'_*] \end{aligned}$$

qed(27)

This implies

$$(28) \ T_* = T'_*.$$

Let $A_* = \bigcup H_*$ and $A'_* = \bigcup H'_*$.

$$(29) \ A_* \restriction (\aleph_{n+1}^V)^2 = A'_* \restriction (\aleph_{n+1}^V)^2.$$

Proof. Note that the map π is the identity on the $*$ -component below \aleph_{n+1}^V . *qed*(29)

$$(30) \text{ For } i < \lambda : A_i \sim A'_i.$$

Proof. Recall $A_i = \bigcup \{q_i \mid (q_*, (b_j, q_j)_{j < \lambda}) \in H\} : [\aleph_0, \aleph_\omega^V) \rightarrow 2$. Since the map π maps the set b_i of linking ordinals to $(b_i \setminus a_i) \cup a'_i$ the linking ordinals in the relevant sets b_i are equal to the linking ordinals in the sets b'_i with possibly finitely many exceptions. This means that the characteristic functions A_i and A'_i will be equal above p_i and p'_i respectively in all cardinal intervals $[\aleph_k, \aleph_{k+1})$ with $k > m$). In other words,

$$(A_i \oplus A'_i) \restriction [\aleph_{m+1}^V, \aleph_\omega^V) \in V.$$

The functions $A_i \restriction \aleph_{m+1}^V$ and $A'_i \restriction \aleph_{m+1}^V$ are determined in the cardinal intervals $[\aleph_k^V, \aleph_{k+1}^V)$ for $k \leq m$ by $p_i \restriction [\aleph_k^V, \aleph_{k+1}^V)$ and $p'_i \restriction [\aleph_k^V, \aleph_{k+1}^V)$ and some cuts $A_*(\xi)$ and $A_*(\xi')$ respectively. Hence $A_i \restriction [\aleph_k^V, \aleph_{k+1}^V), A'_i \restriction [\aleph_k^V, \aleph_{k+1}^V) \in V[A_* \restriction (\aleph_{m+1}^V)^2] = V[A'_* \restriction (\aleph_{m+1}^V)^2]$. Thus

$$(A_i \oplus A'_i) \restriction \aleph_{m+1}^V \in V[H_*] \text{ and } (A_i \oplus A'_i) \restriction [\aleph_{m+1}^V, \aleph_\omega^V) \in V,$$

i.e., $A_i \sim A'_i$. *qed*(30)

This implies immediately that the sequences of equivalence classes agree in both models:

$$(31) \quad \vec{A} = \vec{A'}.$$

$$(32) \quad A_{i_0} = A'_{i_0}, \dots, A_{i_{l-1}} = A'_{i_{l-1}}.$$

Proof. Note that the isomorphism π is the identity at the indices i_0, \dots, i_{l-1} . *qed*(32)

Since $p \Vdash \varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \restriction (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \dots, \dot{A}_{i_{l-1}})$ and $p \in H$ we have

$$V[H] \models \varphi(u, x, T_*, \vec{A}, A_* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}).$$

Since $p' \Vdash \neg\varphi(\check{u}, \check{x}, \sigma, \tau, \dot{A} \restriction (\check{\aleph}_{n+1})^2, \dot{A}_{i_0}, \dots, \dot{A}_{i_{l-1}})$ and $p' \in H'$ we have

$$V[H'] \models \neg\varphi(u, x, T'_*, \vec{A'}, A'_* \restriction P_* \restriction (\aleph_{n+1}^V)^2, A'_{i_0}, \dots, A'_{i_{l-1}}).$$

But the various equalities proved above imply

$$V[H] \models \neg\varphi(u, x, T_*, \vec{A}, A_* \restriction P_* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}),$$

which is the desired contradiction. \blacksquare

5. Wrapping up

We show that the approximation models are mild generic extensions of V .

LEMMA 6: *Let $n < \omega$ and $i_0, \dots, i_{l-1} < \lambda$. Then cardinals are absolute between V and $V[A^* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}]$.*

Proof. Take $p^0 = (p_*^0, (a_i^0, p_i^0)_{i < \lambda}) \in G$ such that $\{i_0, \dots, i_{l-1}\} \subseteq \text{supp}(p^0)$. Since the models $V[A^* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}]$ are monotonely growing with n we may assume that n is large enough such that

$$\forall i \in \text{supp}(p^0) \forall \xi \in a_i^0 : \xi \in \aleph_{n+1}.$$

Since every $A_{i_j} \cap \aleph_{n+1}^V$ can be computed from $A^* \restriction (\aleph_{n+1}^V)^2$, we have

$$\begin{aligned} V[A^* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}] &= \\ &= V[A^* \restriction (\aleph_{n+1}^V)^2, A_{i_0} \restriction [\aleph_{n+1}^V, \aleph_\omega^V), \dots, A_{i_{l-1}} \restriction [\aleph_{n+1}^V, \aleph_\omega^V)]. \end{aligned}$$

Let $P'' = (P'', \supseteq, \emptyset)$ be the forcing

$$P'' = \{r \mid \exists (\delta_m)_{n < m < \omega} (\forall m \in [n+1, \omega) : \delta_m \in [\aleph_m^V, \aleph_{m+1}^V) \wedge r : \bigcup_{n+1 \leq m < \omega} [\aleph_m^V, \delta_m) \rightarrow 2)\},$$

which adjoins COHEN subsets to the \aleph_m 's with $m > n$.

(2) $(A_{i_0} \restriction [\aleph_{n+1}^V, \aleph_\omega^V), \dots, A_{i_{l-1}} \restriction [\aleph_{n+1}^V, \aleph_\omega^V))$ is V -generic for

$$(P'')^l = \underbrace{P'' \times \dots \times P''}_{l \text{ times}}.$$

Proof. Let $D \subseteq (P'')^l$ be dense open, $D \in V$. We have to show that D is met by $(A_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V], \dots, A_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V])$. Let

$$D'' = \{(p_*, (a_i, p_i)_{i < \lambda}) \in Q \mid (p_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V], \dots, p_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V]) \in D\}.$$

This set is dense in P below p^0 : consider $p^1 = (p_*, (a_i^1, p_i^1)_{i < \lambda}) \leq_P (p_*^0, (a_i^0, p_i^0)_{i < \lambda}) = p^0$. Take $(\delta_m)_{n+1 \leq m < \omega}$ such that

$$p_*^1 \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V]^2 : \bigcup_{n+1 \leq m < \omega} [\aleph_m^V, \delta_m]^2 \rightarrow 2.$$

Take $p_{i_0}, \dots, p_{i_{l-1}}$ such that

$$(p_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V], \dots, p_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V]) \in D,$$

and such that $p_{i_0}, \dots, p_{i_{l-1}}$ have the same domains. Through some ordinals in $a_{i_0}^1, \dots, a_{i_{l-1}}^1$, the choice of $p_{i_0}, \dots, p_{i_{l-1}}$ determines some values of p_* by the linking property b) of Definition 1:

$$\forall i < \lambda \forall m < \omega \forall \xi \in a_i \cap [\aleph_m, \aleph_{m+1}) \forall \zeta \in \text{dom}(p_i \setminus p_i^1) \cap [\aleph_m, \aleph_{m+1}) : p_i(\zeta) = p_*(\xi)(\zeta).$$

The independence property implies that the linking sets $a_{i_0}^1, \dots, a_{i_{l-1}}^1$ are pairwise disjoint *above* \aleph_{n+1}^V , i.e., the sets

$$a_{i_0}^1 \cap [\aleph_{n+1}^V, \aleph_\omega^V], \dots, a_{i_{l-1}}^1 \cap [\aleph_{n+1}^V, \aleph_\omega^V]$$

are pairwise disjoint. So the linking requirements can be satisfied simultaneously. Then we can amend the definition of the other components of $p \leq p^1$ and obtain $p \in D''$.

By the genericity of G take $(p_*, (a_i, p_i)_{i < \lambda}) \in D'' \cap G$. Then

$$(p_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V], \dots, p_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V]) \in D$$

with

$$p_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V] \subseteq A_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V], \dots, p_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V] \subseteq A_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V],$$

as required. *qed*(2)

The forcing $(P'')^l$ is $< \aleph_{n+2}$ -closed. $A_* \upharpoonright (\aleph_{n+1}^V)^2$ is V -generic for the forcing

$$P_* \upharpoonright (\aleph_{n+1}^V)^2 = \{r \upharpoonright (\aleph_{n+1}^V)^2 \mid r \in P_*\}.$$

By the GCH in V , $\text{card}(P_* \upharpoonright (\aleph_{n+1}^V)^2) = \aleph_{n+1}$. Hence every dense subset of $P_* \upharpoonright (\aleph_{n+1}^V)^2$ which is in $V[A_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V], \dots, A_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V]]$ is already an element of V . Thus $A_* \upharpoonright (\aleph_{n+1}^V)^2$ is $V[A_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V], \dots, A_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V]]$ -generic for $P_* \upharpoonright (\aleph_{n+1}^V)^2$. By standard properties of product forcing,

$$A_* \upharpoonright (\aleph_{n+1}^V)^2 \times (A_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V], \dots, A_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V])$$

is generic for the forcing $P_* \upharpoonright (\aleph_{n+1}^V)^2 \times (P'')^l$. This forcing is canonically isomorphic to the initial forcing P_0 . By Lemma 1, cardinals are preserved between V and $V[A_* \upharpoonright (\aleph_{n+1}^V)^2, A_{i_0} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V], \dots, A_{i_{l-1}} \upharpoonright [\aleph_{n+1}^V, \aleph_\omega^V]]$. ■

LEMMA 7: Cardinals are absolute between N and V , and in particular $\kappa = \aleph_\omega^V = \aleph_\omega^N$.

Proof. If not, then there is a function $f \in N$ which collapses a cardinal in V . By Lemma 5, f is an element of some model $V[A_* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}]$ as above. But this contradicts Lemma 6. ■

LEMMA 8: *GCH holds in N below \aleph_ω .*

Proof. If $X \subseteq \aleph_n$ and $X \in N$ then X is an element of some model $V[A_* \restriction (\aleph_{n+1}^V)^2, A_{i_0}, \dots, A_{i_{l-1}}]$ as above. Since $A_{i_0}, \dots, A_{i_{l-1}}$ do not adjoin new subsets of \aleph_n we have that

$$X \in V[A_* \restriction (\aleph_{n+1}^V)^2].$$

Hence $\mathcal{P}(\aleph_n^V) \cap N \in V[A_* \restriction (\aleph_{n+1}^V)^2]$. The proof of Lemma 1 shows that GCH holds in $V[A_* \restriction (\aleph_{n+1}^V)^2]$. Hence there is a bijection $\mathcal{P}(\aleph_n^V) \cap N \leftrightarrow \aleph_{n+1}^V$ in $V[A_* \restriction (\aleph_{n+1}^V)^2]$ and hence in N . ■

6. Discussion and Remarks

The above construction straightforwardly generalises to other cardinals κ of cofinality ω . In that extension, cardinals $\leq \kappa$ are preserved, GCH holds below κ , and there is a surjection from $\mathcal{P}(\kappa)$ onto some arbitrarily high cardinal λ . To work with singular cardinals κ of *uncountable* cofinality, finiteness properties in the construction have to be replaced by the property of being of cardinality $< \text{cof}(\kappa)$. This yields results like the following choiceless violation of SILVER's theorem [6].

THEOREM 2: *Let V be any ground model of ZFC + GCH and let λ be some cardinal in V . Then there is a cardinal preserving model $N \supseteq V$ of the theory ZF + “GCH holds below \aleph_{ω_1} ” + “there is a surjection from $\mathcal{P}(\aleph_{\omega_1})$ onto λ ”. Moreover, the axiom of dependent choices DC holds in N .*

Note that in [5], SAHARON SHELAH studied uncountably singular cardinal arithmetic under DC, without assuming AC. The “local” GCH below \aleph_{ω_1} in the conclusion of the above Theorem cannot be changed to the property $\text{card}(\bigcup_{\alpha < \aleph_{\omega_1}} \mathcal{P}(\alpha)) = \aleph_{\omega_1}$ since Theorem 4.6 of [5] basically implies that then $\mathcal{P}(\aleph_{\omega_1})$ would be wellorderable of ordertype $\geq \lambda$. By results of [1] an *injective* failure of SCH with big λ has high consistency strength. But here we are working without assuming any large cardinals.

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